Matematica în Bucovina. Concursul Internaţional de matematică „Memorialul *David Hrimiuc*” ediţia a XIII - a, 4 noiembrie – 6 noiembrie 2016

Clasa a VI - a

Barem de corectare

1. (7p) Arătaţi că $n = 6^1 + 6^2 + 6^3 + \cdots + 6^{2016}$ este divizibil cu 43, 111, 259 şi 777.

Soluţie. $2016 = 2^5 \cdot 3^2 \cdot 7 \Rightarrow 3/2016\ ,\ 4/2016$ **(0,5p)**

$$n = \left(6^1 + 6^2 + 6^3\right) + \left(6^4 + 6^5 + 6^6\right) + \left(6^7 + 6^8 + 6^9\right) + \cdots \left(6^{2014} + 6^{2015} + 6^{2016}\right) = 6^1 \cdot \left(1 + 6 + 6^2\right) +$$

$$+ 6^4 \cdot \left(1 + 6 + 6^2\right) + 6^7 \cdot \left(1 + 6 + 6^2\right) + \cdots + 6^{2014} \cdot \left(1 + 6 + 6^2\right) = 43 \cdot \left(6^1 + 6^4 + 6^7 + \cdots + 6^{2014}\right) \Rightarrow 43/n \quad \textbf{(2p)}$$

$$n = \left(6^1 + 6^2 + 6^3 + 6^4\right) + \left(6^5 + 6^6 + 6^7 + 6^8\right) + \cdots + \left(6^{2013} + 6^{2014} + 6^{2015} + 6^{2016}\right) = 6^1 \cdot \left(1 + 6 + 6^2 + 6^3\right) +$$

$$+ 6^5 \cdot \left(1 + 6 + 6^2 + 6^3\right) + \cdots + 6^{2013} \cdot \left(1 + 6 + 6^2 + 6^3\right) = 259 \cdot \left(6^1 + 6^5 + \cdots + 6^{2013}\right) = 7 \cdot 37 \cdot \left(6^1 + 6^5 + \cdots + 6^{2013}\right) \Rightarrow$$

$\Rightarrow 259/n\ ,\ 7/n$ şi $37/n$ **(2,5p)**

SAU:

$$n = \left(6^1 + 6^2\right) + \left(6^3 + 6^4\right) + \cdots + \left(6^{2015} + 6^{2016}\right) = 6^1\left(1 + 6\right) + 6^3\left(1 + 6\right) + \cdots + 6^{2015}\left(1 + 6\right) =$$

$$= 7\left(6^1 + 6^3 + \cdots + 6^{2015}\right) \Rightarrow 7/n \quad \textbf{(1p)}$$

$$n = \left(6^1 + 6^3\right) + \left(6^2 + 6^4\right) + \left(6^5 + 6^7\right) + \left(6^6 + 6^8\right) + \cdots + \left(6^{2013} + 6^{2015}\right) + \left(6^{2014} + 6^{2016}\right) =$$

$$= 6^1\left(1 + 6^2\right) + 6^2\left(1 + 6^2\right) + 6^5\left(1 + 6^2\right) + 6^6\left(1 + 6^2\right) + \cdots + 6^{2013}\left(1 + 6^2\right) + 6^{2014}\left(1 + 6^2\right) =$$

$$= 37\left(6^1 + 6^2 + 6^5 + 6^6 + \cdots + 6^{2009} + 6^{2010} + 6^{2013} + 6^{2014}\right) \Rightarrow 37/n \quad \textbf{(1p)}$$

$$\left.\begin{array}{r} 7/n \\ 37/n \\ 7\ ,\ 37 - nr.\ prime \end{array}\right\} \Rightarrow 7 \cdot 37/n \Rightarrow 259/n \quad \textbf{(0,5p)}$$

Acum:
$$\left.\begin{array}{r} 6/n \Rightarrow 3/n \\ 37/n \\ 3\ ,\ 37 - nr.\ prime \end{array}\right\} \Rightarrow 3 \cdot 37/n \Rightarrow 111/n \quad \textbf{(1p)}$$

$$\left.\begin{array}{r} 3/n \\ 7/n \\ 37/n \\ 3\ ,\ 7\ ,\ 37 - nr.\ prime \end{array}\right\} \Rightarrow 3 \cdot 7 \cdot 37/n \Rightarrow 777/n \quad \textbf{(1p)}$$

2. (7p) Determinaţi numărul $\overline{abcd}$ astfel încât $\frac{\overline{ab}}{5} = \frac{\overline{cd}}{4} = \frac{\overline{ca}}{3} = \frac{\overline{db}}{15}$.

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Soluţia 1.

$$\left.\begin{array}{l} \left.\begin{array}{r} \frac{\overline{ab}}{5} = \frac{\overline{cd}}{4} \Rightarrow 4 \cdot \overline{ab} = 5 \cdot \overline{cd} \quad (0,5p) \quad \Rightarrow 5/4 \cdot \overline{ab} \quad (1p) \\ (5,4) = 1 \end{array}\right\} \Rightarrow 5/\overline{ab} \quad (0,5p) \Rightarrow \frac{\overline{ab}}{5} \in \mathbb{N} \\ \frac{\overline{ab}}{5} = \frac{\overline{cd}}{4} = \frac{\overline{db}}{15} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow \overline{db}\in\{15,30,45,60,75,90\}\ (1p)\\ \dfrac{\overline{cd}}{4}\in\mathbb{N}\ \Rightarrow d-nr.\ par\quad (0,5p)\end{array}\right\}\Rightarrow \overline{db}\in\{45,60\}\quad (1p)$$

$$\left.\begin{array}{l}\overline{db}=45\\ \dfrac{\overline{cd}}{4}=\dfrac{\overline{db}}{15}\end{array}\right\}\Rightarrow \overline{c4}=12\quad fals!\quad (0,5p)$$

$$\left.\begin{array}{r}\overline{db}=60\\ \dfrac{\overline{ab}}{5}=\dfrac{\overline{cd}}{4}=\dfrac{\overline{db}}{15}\end{array}\right\}\Rightarrow\begin{cases}\overline{c6}=16\\ \overline{a0}=20\end{cases}\ (1p)\ \Rightarrow\ \begin{cases}c=1\\ a=2\end{cases}\ (0,5p)\ \Rightarrow\ \overline{abcd}=2016\quad (0,5p)$$

Soluţia 2.

$$\left.\begin{array}{r}\dfrac{\overline{ab}}{5}=\dfrac{\overline{ca}}{3}\ \Rightarrow\ 3\cdot\overline{ab}=5\cdot\overline{ca}\ (1p)\ \Rightarrow\ 3(10a+b)=5(10c+a)\ (1p)\ \Rightarrow\ 3b=25(2c-a)\\ b-cifră\ \Rightarrow\ 0\le 3b\le 27\quad (0,5p)\end{array}\right\}\Rightarrow$$

$$\Rightarrow\quad b=0\ \text{şi}\ a=2c\quad (1p)$$

$$\left.\begin{array}{r}\dfrac{\overline{ab}}{5}=\dfrac{\overline{db}}{15}\overset{b=0}{\Rightarrow}\ 30a=10d\ \Rightarrow\ d=3a\ (1p)\overset{a=2c}{\Rightarrow}\ d=6c\quad (1p)\\ c,d-cifre\ nenule\end{array}\right\}\Rightarrow c=1,\ d=6,\ a=2\ (1p)\Rightarrow$$

$$\Rightarrow\quad \overline{abcd}=2016\quad (0,5p)$$

3. a) (3p) Scrieţi numărul $n=5^{2016}-5^{2015}+5^{2014}-5^{2013}$ ca o sumă de cinci pătrate perfecte nenule.

Soluţie. $n=5^{2016}-5^{2015}+5^{2014}-5^{2013}\overset{(0,5p)}{=}5^{2015}(5-1)+5^{2013}(5-1)=4\cdot5^{2013}\left(5^2+1\right)\overset{(0,5p)}{=}$

$=4\cdot5^{2012}\cdot5\cdot26\overset{(0,5p)}{=}4\cdot5^{2012}\cdot130\overset{(0,5p)}{=}\left(2\cdot5^{1006}\right)^2\left(1^2+2^2+3^2+4^2+10^2\right)\overset{(0,5p)}{=}\left(2\cdot5^{1006}\right)^2+\left(4\cdot5^{1006}\right)^2+$

$+\left(6\cdot5^{1006}\right)^2+\left(8\cdot5^{1006}\right)^2+\left(20\cdot5^{1006}\right)^2\overset{(0,5p)}{=}a^2+b^2+c^2+d^2+e^2$.

b) (4p) Determinaţi numărul natural n pentru care numerele $n-7$, $n-3$, $n-1$, $n+3$, $n+5$, $n+9$, $n+15$ sunt simultan numere prime.

Soluţie. Notăm $n-7=p\Rightarrow n=p+7$, $p\ge 2$. Condiţia din ipoteză se transpune prin: p, $p+4$, $p+6$, $p+10$, $p+12$, $p+16$, $p+22$ sunt simultan numere prime. Putem avea:

$\left.\begin{array}{l}p=7k,k\ge1\\ p-nr.\ prim\end{array}\right\}\Rightarrow k=1\Rightarrow p=7$ .Soluţie, pentru că 7, 11, 13, 17, 19, 23, 29 sunt numere prime. **(0,5p)**

$p=7k+1$, $k\in\mathbb{N}^*\ \Rightarrow p+6=7(k+1)\ne nr.\ prim$ Fals! **(0,5p)**

$p=7k+2$, $k\in\mathbb{N}\ \Rightarrow p+12=7(k+2)\ne nr.\ prim$ Fals! **(0,5p)**

$p=7k+3$, $k\in\mathbb{N}\ \Rightarrow\begin{cases}k\in\mathbb{N}^*\Rightarrow\ p+4=7(k+2)\ne nr.\ prim\\ k=0\Rightarrow p=3\Rightarrow p+6=9\ne nr.\ prim\end{cases}$ Fals! **(0,5p)**

$p=7k+4$, $k\in\mathbb{N}\ \Rightarrow p+10=7(k+2)\ne nr.\ prim$ Fals! **(0,5p)**

$p=7k+5$, $k\in\mathbb{N}\ \Rightarrow p+16=7(k+3)\ne nr.\ prim$ Fals! **(0,5p)**

$p=7k+6$, $k\in\mathbb{N}\ \Rightarrow p+22=7(k+4)\ne nr.\ prim$ Fals! **(0,5p)**

Numărul căutat este $n=14$. **(0,5p)**

4. **(7p)** Pe o dreaptă d se iau punctele A şi B. Fie punctul M mijlocul segmentului $[AB]$ şi punctul N astfel încât $N \in AB$ şi $N \notin [AB]$. Dacă punctul P este mijlocul segmentului $[MN]$, arătaţi că $AN + BN = 4MP$.

Soluţie.

$M - mijl.[AB] \Rightarrow AM = MB = \dfrac{AB}{2} \Rightarrow AB = 2 \cdot AM = 2 \cdot MB$ (1) **(1p)**

$P - mijl.[MN] \Rightarrow MP = \dfrac{MN}{2} \Rightarrow MN = 2 \cdot MP$ (2) **(1p)**

$N \in AB$, $N \notin [AB] \Rightarrow A \in (NB)$ *sau* $B \in (AN)$ **(1p)**

Cazul I. $A \in (NB)$

(0,5p)

$AN + BN = AN + AN + AB = 2AN + AB \overset{(1)}{=} 2AN + 2AM = 2(AN + AM) = 2NM \overset{(2)}{=} 4MP$ **(1,5p)**

Cazul II. $B \in (AN)$

(0,5p)

$AN + BN = AB + BN + BN = AB + 2BN \overset{(1)}{=} 2MB + 2BN = 2(MB + BN) = 2NM \overset{(2)}{=} 4MP$ **(1,5p)**

Notă: **Orice altă rezolvare corectă se punctează corespunzător.**